On space-times admitting shear-free, irrotational, geodesic null congruences

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Abstract

Space-times admitting a shear-free, irrotational, geodesic null congruence are studied. Attention is focused on those space-times in which the gravitational field is a combination of a perfect fluid and null radiation.

1 Introduction

In this article we wish to extend earlier work on shear-free, irrotational and geodesic (SIG) timelike and spacelike congruences [1, 2] to SIG *null* congruences. The fact that we are dealing with null congruences means that we have to approach the problem in a completely different way; we must make extensive use of the Newman-Penrose formalism.

Thus, we wish to study a congruence of curves whose tangent vector $\mathbf{k}$ is null and geodesic. Hence, we have a family of null geodesics $x^a = x^a(y^\alpha, v)$, where y^α distinguishes the different geodesics, and v is the affine parameter along a fixed geodesic. The null tangent vector is $k^a = \frac{\partial x^a}{\partial v}$, and satisfies $k^a{}_{;b}k^b = 0$. The spin coefficients are defined in [3], where $\rho = -(\theta + i\omega)$ is called the complex divergence and σ is the complex shear. The geodesic condition implies that the spin coefficient κ vanishes and $\epsilon + \bar{\epsilon} = 0$ follows from the choice of an affine parameter along the congruence. The congruence is said to be shear-free if $\sigma = 0$. Also, from the relation $k_{[a;b}k_{c]} = (\bar{\rho} - \rho)\bar{m}_{[a}m_b k_{c]}$ [4], it follows that $w = 0$ (i.e., zero twist) is a necessary and sufficient condition for $\mathbf{k}$ to be hypersurface orthogonal.

First we shall briefly review some of the results of relevance to this work. Goldberg and Sachs [5] proved that if a gravitational field contains a shear-free, geodesic, null congruence $\mathbf{k}$, then $\kappa = \sigma = 0$, and if

$$R_{ab}k^a k^b = R_{ab}k^a m^b = R_{ab}m^a m^b = 0 \ , \tag{1}$$

then the field is algebraically special (i.e., $\Psi_0 = \Psi_1 = 0$), and $\mathbf{k}$ is a degenerate eigendirection. In addition, a vacuum metric is algebraically special if and only if it contains a shear-free geodesic null congruence.

A space-time admits a geodesic, shear-free, twist-free ($\kappa = \sigma = \omega = 0$) and diverging ($\rho = \bar{\rho} = \theta = -1/r$) null congruence $\mathbf{k}$, and satisfies (1), if and only if the metric can be written in the form

$$ds^2 = 2r^2 P^{-2}(z, \bar{z}, u)dz d\bar{z} - 2du dr - 2H(z, \bar{z}, r, u)du^2 \ . \tag{2}$$

Robinson-Trautman models [6] with this metric have been found for vacuum, Einstein-Maxwell and pure radiation fields with or without a cosmological constant [3].

For geodesic null vector fields we have that $(\theta+i\omega)_{,a}k^a+(\theta+i\omega)^2+\sigma\bar{\sigma}=-R_{ab}k^ak^b/2$. Therefore, in the non-diverging case (i.e., $\rho=-(\theta+i\omega)=0$), if the energy condition $T_{ab}k^ak^b\geq 0$ is satisfied, it follows that $\sigma=0=R_{ab}k^ak^b$. Thus, non-twisting (and therefore geodesic) and non-expanding null congruences must be shear-free. Hence, the space-time is algebraically special, and it corresponds to vacuum, Einstein-Maxwell, and pure radiation field. Perfect fluid solutions violate $R_{ab}k^ak^b=0$ unless $\mu+p=0$. This class of solutions has been studied by Kundt [7].

Another important case corresponds to the Kerr-Schild metric, which is given by $g_{ab}=\eta_{ab}-2\phi k_ak_b$. The null vector $\mathbf{k}$ of a Kerr-Schild metric is geodesic if and only if the energy-momentum tensor obeys the condition $T_{ab}k^ak^b=0$, and then $\mathbf{k}$ is a multiple principal null direction of the Weyl tensor and the space-time is algebraically special. The general properties of the Kerr-Schild metrics and their applications to vacuum, Einstein-Maxwell, and pure radiation space-times can be found in [3].

Finally, we note the algebraically special perfect fluid space-times corresponding to the generalized Robinson-Trautman solutions investigated by Wainwright [8]. They are characterized by a multiple null eigenvector $\mathbf{k}$ of the Weyl tensor which is geodesic, shear-free, and twist-free but expanding (i.e., $\Psi_o=\Psi_1=0$, $\kappa=\sigma=\omega=0$, $\rho=\bar{\rho}\neq 0$), and the four-velocity obeys $u_{[a;b}u_{c]}=0$, $k_{[c}k_{a];b}u^b=0$. The line-element of the space-time can be written in the form

$$ds^2=-\frac{1}{2}\chi^2(r,u)P^{-2}(z,\bar{z},u)dzd\bar{z}+2du(dr-Udu)\ , \tag{3}$$

with

$$U=r(\ln P)_{,u}+U^0(z,\bar{z},u)+S(r,u)\ ,\quad \chi_{,r}\neq 0\ ,\quad \frac{\chi_{,rr}}{\chi}\leq 0\ . \tag{4}$$

In this case no dust solutions nor solutions of Petrov types III and N are possible.

2 Analysis

Let us consider space-times admitting a shear-free, irrotational, geodesic null congruence in which the source of the gravitational field is a *combination of a perfect fluid and null radiation*, so that the energy-momentum tensor has the form

$$T_{ab}=(\mu+p)u_au_b-pg_{ab}+\phi^2k_ak_b\ , \tag{5}$$

where u^a is the four-velocity of the fluid, μ and p are the density and the pressure of the fluid, respectively, and $\mathbf{k}$ is a null vector. The null radiation is geodesic, twist-free, and shear-free, and defines the null congruence. Wainwright [8] proved that for a space-time in which there exists a SIG null congruence, coordinates can be chosen so that the metric takes on the simplified form (3) with $u=x^1$, $r=x^2$, $z=x^3+ix^4$, the tangent field of the null congruence is given by $k^a=\delta^a_2$, $k_a=\delta^1_a$, and we can introduce the null tetrad

$$k^a=\delta^a_r\ ,\quad l^a=\delta^a_u+U\delta^a_r\ ,\quad m^a=P\chi^{-1}(\delta^a_3+i\delta^a_4)\ , \tag{6}$$

$$k_a=\delta^u_a\ ,\quad l_a=-U\delta^u_a+\delta^r_a\ ,\quad m_a=P^{-1}\chi(\delta^3_a+i\delta^4_a)/2\ . \tag{7}$$

With the sign convention used here we have that $u^au_a=k^al_a=1=-m^a\bar{m}_a$. Note that the null radiation is everywhere tangent to the repeated null congruence of the space-time.

First, since $\Phi_{01}\equiv-\frac{1}{2}R_{ab}k^am^b=0$, we conclude that the four-velocity satisfies $u^am_a=0$, and hence it can be expressed in terms of the null tetrad by

$$u^a=\frac{1}{\sqrt{2}B}(B^2k^a+l^a)\quad\text{and}\quad u_a=\frac{1}{\sqrt{2}B}[(B^2-U)\delta^u_a+\delta^r_a]\ , \tag{8}$$

for some function B. The conditions $\Phi_{02} \equiv -\frac{1}{2}R_{ab}m^a m^b = 0$ and $\Phi_{12} \equiv -\frac{1}{2}R_{ab}m^a l^b = 0$ are satisfied identically. The non-zero components of the Ricci tensor are

$$\Phi_{00} \equiv -\frac{1}{2}(R_{ab} - \frac{1}{4}Rg_{ab})k^a k^b = \frac{1}{2}(\mu + p)(\mathbf{k}\cdot\mathbf{u})^2 \ , \tag{9}$$

$$\Phi_{11} \equiv -\frac{1}{4}(R_{ab} - \frac{1}{4}Rg_{ab})(k^a l^b + m^a \bar{m}^b) = \frac{1}{4}(\mu + p)(\mathbf{k}\cdot\mathbf{u})(\mathbf{l}\cdot\mathbf{u}) \ , \tag{10}$$

$$\Phi_{22} \equiv -\frac{1}{2}(R_{ab} - \frac{1}{4}Rg_{ab})l^a l^b = \frac{1}{2}(\mu + p)(\mathbf{l}\cdot\mathbf{u})^2 + \frac{1}{2}\phi^2 \ . \tag{11}$$

In addition, since $\mathbf{k}\cdot\mathbf{u} = \frac{1}{\sqrt{2}B}$ and $\mathbf{l}\cdot\mathbf{u} = \frac{1}{\sqrt{2}}B$ implies $\mathbf{l}\cdot\mathbf{u} = B^2(\mathbf{k}\cdot\mathbf{u})$, we obtain

$$B^2\Phi_{00} \quad = \quad 2\Phi_{11} \ , \tag{12}$$

$$B^4\Phi_{00} \quad = \quad \Phi_{22} - \frac{1}{2}\phi^2 \ . \tag{13}$$

If we now assume that the fluid is non-rotating, then $B^2 = U + F(r,u)$, and the compatibility condition (12) can be written as

$$(U + F)\Phi_{00} = 2\Phi_{11} \ . \tag{14}$$

On differentiating this equation successively with respect to z and r, we obtain the restriction

$$(\chi^2)_{,rrr}[U^0{}_{,z} + r(\ln P)_{,uz}] = 0 \ . \tag{15}$$

There are consequently two different cases to consider.

In the first case $U^0{}_{,z} + r(\ln P)_{,uz} = 0$, which is equivalent to $U^0{}_{,z} = (\ln P)_{,uz} = 0$, so that $P = P(z, \bar{z})$ and $U^0 = U^0(u)$. Obviously, the solutions admit a multiply transitive group of motions, G_3, acting on the 2-spaces $r =$const, $u =$const, of constant curvature, and belong to class II of Stewart and Ellis [9]. The metric (3) can then be rewritten as

$$ds^2 = -\chi^2(r,u)\frac{2dzd\bar{z}}{(1 + \frac{k}{2}z\bar{z})^2} + 2du(dr - U(r,u)du) \ . \tag{16}$$

The non-zero Ricci components are given by

$$\Phi_{00} = -\frac{\chi_{,rr}}{\chi} \ , \tag{17}$$

$$\Phi_{11} = \frac{\chi_{,r}\chi_{,u}}{2\chi^2} + \frac{(\chi_{,r})^2 U}{2\chi^2} - \frac{U_{,rr}}{4} + \frac{k}{4\chi^2} \ , \tag{18}$$

$$\Phi_{22} = \frac{\chi_{,u}U_{,r}}{\chi} - \frac{\chi_{,uu}}{\chi} - 2\frac{\chi_{,ur}U}{\chi} - \frac{\chi_{,r}U_{,u}}{\chi} - \frac{\chi_{,rr}U^2}{\chi} \ , \tag{19}$$

and the Ricci scalar is given by

$$\frac{R}{2} = 12\Lambda = 4\frac{\chi_{,r}U_{,r}}{\chi} + 2\frac{\chi_{,r}\chi_{,u}}{\chi^2} + 2\frac{(\chi_{,r})^2 U}{\chi^2} + 4\frac{\chi_{,ur}}{\chi} + U_{,rr} + 4\frac{\chi_{,rr}U}{\chi} + \frac{k}{\chi^2} \ . \tag{20}$$

Hence, the metric (16) can be interpreted as pure radiation plus a perfect fluid where μ and p are given by

$$\mu = \frac{R}{4} + 6\Phi_{11} \ , \quad p = -\frac{R}{4} + 2\Phi_{11} \ , \tag{21}$$

u_a is determined by (8) with $B^2 = 2\Phi_{11}/\Phi_{00}$, and ϕ^2 is given by

$$\phi^2 = 2\left(\Phi_{22} - 4\frac{\Phi_{11}^2}{\Phi_{00}}\right) \ . \tag{22}$$

In the second case (i.e., ${\chi^2}_{,rrr} = 0$) two possibilities arise:

$$(i) \qquad \chi^2 = \epsilon r, \qquad \epsilon = \pm 1 \tag{23}$$

$$(ii) \qquad \chi^2 = \epsilon(r^2 - k^2), \quad k = const\ . \tag{24}$$

In both subcases $\chi = \chi(r)$, and they can be written together as $\chi^2 = ar^2 + 2br + c$, with a, b, c taken to be appropriate constants. From equation (14) we obtain

$$aU^0 - b(\ln P)_{,u} + K = G(u)\ , \tag{25}$$

and

$$\frac{1}{2}[\chi^2 S_{,r} - S(\chi^2)_{,r}]_{,r} + \frac{F\Sigma}{\chi^2} = G(u)\ , \tag{26}$$

where $K \equiv 4P^2(\ln P)_{z\bar{z}}$, $\Sigma \equiv b^2 - ac$, and $G(u)$ is an arbitrary function of u.

Subcase (i): $a = c = 0$, $b = \epsilon/2$. Integrating equation (26) we see that S can be written in the form

$$S = rh(u) + 2\epsilon G(u) r \ln r - f(u) - \frac{1}{2} r \int \frac{dr}{r^2} \int^r \frac{d\hat{r}}{\hat{r}} F(\hat{r}, u)\ , \tag{27}$$

where $h(u)$ and $f(u)$ are arbitrary functions of u.

Subcase (ii): $a = \epsilon$, $b = 0$, $c = -\epsilon k^2$, $\Sigma = k^2$. We obtain

$$S = -\epsilon G(u) + f(u)\chi^2 \int \frac{dr}{\chi^4} + h(u)\chi^2 - 2k^2\chi^2 \int \frac{dr}{\chi^4} \int^r \frac{d\hat{r}}{\chi^2(\hat{r})} F(\hat{r}, u)\ . \tag{28}$$

Therefore, the metric (3) with $\chi(r)$ given by (23) or (24), $S(r, u)$ given by (27) or (28), and $P(z, \bar{z}, u)$ satisfying (25) can be interpreted as pure radiation plus a perfect fluid, in which the four-velocity is determined by (8) and ϕ^2, μ and p are determined by (21) and (22), respectively.

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